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CITATION:

Kawashima, Shuichi ...[et al]. A Hardy type inequality and application to the stability of degenerate stationary waves (Mathematical Analysis in Fluid and Gas Dynamics). 数理解析研究所講究録 2009, 1631: 34-46

ISSUE DATE:

2009-02

URL:

<http://hdl.handle.net/2433/140413>

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# A Hardy type inequality and application to the stability of degenerate stationary waves

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## 1 Introduction

This note is a survey of our joint paper [2] on the stability problem of degenerate stationary waves for viscous conservation laws in the half space  $x > 0$ :

$$\begin{aligned} u_t + f(u)_x &= u_{xx}, \\ u(0, t) &= -1, \quad u(x, 0) = u_0(x). \end{aligned} \tag{1.1}$$

Here  $u_0(x) \rightarrow 0$  as  $x \rightarrow \infty$ , and  $f(u)$  is a smooth function satisfying

$$f(u) = \frac{1}{q}(-u)^{q+1}(1 + g(u)), \quad f''(u) > 0 \quad \text{for } -1 \leq u < 0, \tag{1.2}$$

where  $q$  is a positive integer (degeneracy exponent) and  $g(u) = O(|u|)$  for  $u \rightarrow 0$ . Notice that  $1 + g(u) > 0$  for  $-1 \leq u \leq 0$ . It is known that the corresponding stationary problem

$$\begin{aligned} \phi_x &= f(\phi), \\ \phi(0) &= -1, \quad \phi(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty, \end{aligned} \tag{1.3}$$

admits a unique solution  $\phi(x)$  (called *degenerate stationary wave*) which verifies  $\phi(x) \sim -(1+x)^{-1/q}$ . In particular, we have  $\phi(x) = -(1+x)^{-1/q}$  when  $g(u) \equiv 0$ .

To discuss the stability of the degenerate stationary wave  $\phi(x)$ , it is convenient to introduce the perturbation  $v$  by  $u(x, t) = \phi(x) + v(x, t)$  and rewrite the problem (1.1) as

$$\begin{aligned} v_t + (f(\phi + v) - f(\phi))_x &= v_{xx}, \\ v(0, t) &= 0, \quad v(x, 0) = v_0(x), \end{aligned} \tag{1.4}$$

where  $v_0(x) = u_0(x) - \phi(x)$ , and  $v_0(x) \rightarrow 0$  as  $x \rightarrow \infty$ . The stability of degenerate stationary waves has been studied recently in [14, 2]. The paper [14] proved the following stability result: If the initial perturbation  $v_0(x)$  is in the weighted space  $L_\alpha^2$ , then the perturbation  $v(x, t)$  decays in  $L^2$  at the rate  $t^{-\alpha/4}$  as  $t \rightarrow \infty$ , provided that  $\alpha < \alpha_*(q)$ , where

$$\alpha_*(q) := (q + 1 + \sqrt{3q^2 + 4q + 1})/q.$$

The decay rate  $t^{-\alpha/4}$  obtained in [14] would be optimal but the restriction  $\alpha < \alpha_*(q)$  was not very sharp. This restriction has been relaxed to  $\alpha < \alpha_c(q) := 3 + 2/q$  in our joint paper [2] by employing the space-time weighted energy method in [14] and by applying a Hardy type inequality with the best possible constant. Notice that  $\alpha_*(q) < \alpha_c(q)$ . This new stability result will be reviewed in this note.

It is interesting to note that a similar restriction on the weight is imposed also for the stability of degenerate shock profiles (see [9]). We remark that our stability result for degenerate stationary waves is completely different from those for non-degenerate case. In fact, for non-degenerate stationary waves, we have the better decay rate  $t^{-\alpha/2}$  for the perturbation without any restriction on  $\alpha$ . See [4, 5, 13, 15] for the details. See also [6, 8, 10] for the related stability results for stationary waves.

To check the validity of our restriction  $\alpha < \alpha_c(q) := 3 + 2/q$ , it is important to discuss the dissipativity of the following linearized operator associated with (1.4):

$$Lv = v_{xx} - (f'(\phi)v)_x. \tag{1.5}$$

In a simpler situation including the case  $g(u) \equiv 0$  in (1.2), we showed in [2] that the operator  $L$  is uniformly dissipative in  $L_\alpha^2$  for  $\alpha < \alpha_c(q)$  but can not be dissipative for  $\alpha > \alpha_c(q)$ . This suggests that the exponent  $\alpha_c(q)$  is the critical exponent of the stability problem of degenerate stationary waves. This result on the characterization of the dissipativity of  $L$  is an improvement on the previous one in [14] and has been established again by using a Hardy

type inequality with the best possible constant. This result will be also reviewed in this note.

**Notations.** For  $1 \leq p \leq \infty$  and a nonnegative integer  $s$ ,  $L^p$  and  $W^{s,p}$  denote the usual Lebesgue space on  $\mathbb{R}_+ = (0, \infty)$  and the corresponding Sobolev space, respectively. When  $p = 2$ , we write  $H^s = W^{s,2}$ . We introduce weighted spaces. Let  $w = w(x) > 0$  be a weight function defined on  $[0, \infty)$  such that  $w \in C^0[0, \infty)$ . Then, for  $1 \leq p < \infty$ , we denote by  $L^p(w)$  the weighted  $L^p$  space on  $\mathbb{R}_+$  equipped with the norm

$$\|u\|_{L^p(w)} := \left( \int_0^\infty |u(x)|^p w(x) dx \right)^{1/p}. \quad (1.6)$$

The corresponding weighted Sobolev space  $W^{s,p}(w)$  is defined by  $W^{s,p}(w) = \{u \in L^p(w); \partial_x^k u \in L^p(w) \text{ for } k \leq s\}$  with the norm  $\|\cdot\|_{W^{s,p}(w)}$ . Also, we denote by  $W_0^{1,p}(w)$  the completion of  $C_0^\infty(\mathbb{R}_+)$  with respect to the norm

$$\|u\|_{W_0^{1,p}(w)} := \|\partial_x u\|_{L^p(w)} = \left( \int_0^\infty |\partial_x u(x)|^p w(x) dx \right)^{1/p}. \quad (1.7)$$

When  $p = 2$ , we write  $H^s(w) = W^{s,2}(w)$  and  $H_0^1(w) = W_0^{1,2}(w)$ . In the special case where  $w = (1+x)^\alpha$  with  $\alpha \in \mathbb{R}$ , these weighted spaces are abbreviated as  $L_\alpha^p$ ,  $W_\alpha^{s,p}$ ,  $W_{\alpha,0}^{1,p}$ ,  $H_\alpha^s$  and  $H_{\alpha,0}^1$ , respectively.

## 2 Hardy type inequality

Our Hardy type inequality used in [2] is a simple modification of the original Hardy's inequality introduced in [1, 7] (see also [12]).

**Proposition 2.1.** *Let  $\psi \in C^1[0, \infty)$  and assume either*

- (1)  $\psi > 0$ ,  $\psi_x > 0$  and  $\psi(x) \rightarrow \infty$  for  $x \rightarrow \infty$ ; or
- (2)  $\psi < 0$ ,  $\psi_x > 0$  and  $\psi(x) \rightarrow 0$  for  $x \rightarrow \infty$ .

*Then we have*

$$\int_0^\infty v^2 \psi_x dx \leq 4 \int_0^\infty v_x^2 \psi^2 / \psi_x dx \quad (2.1)$$

*for  $v \in C_0^\infty(\mathbb{R}_+)$  and hence for  $v \in H_0^1(w)$  with  $w = \psi^2 / \psi_x$ . Here 4 is the best possible constant, and there is no function  $v \in H_0^1(w)$ ,  $v \neq 0$ , which attains the equality in (2.1).*

*Proof.* The proof is quite simple. Let  $v \in C_0^\infty(\mathbb{R}_+)$ . A simple calculation gives

$$\begin{aligned} (v^2 \psi)_x &= v^2 \psi_x + 2v v_x \psi \\ &= \frac{1}{2} v^2 \psi_x + \frac{1}{2} (v + 2v_x \psi / \psi_x)^2 \psi_x - 2v_x^2 \psi^2 / \psi_x. \end{aligned} \quad (2.2)$$

Integrating (2.2) in  $x$ , we obtain

$$\int_0^\infty v^2 \psi_x dx + \int_0^\infty (v + 2v_x \psi / \psi_x)^2 dx = 4 \int_0^\infty v_x^2 \psi^2 / \psi_x dx, \quad (2.3)$$

which gives the desired inequality (2.1). It follows from (2.3) that the equality in (2.1) holds if and only if  $v + 2v_x \psi / \psi_x \equiv 0$ . But we find that such a  $v$  in  $H_0^1(w)$  must be  $v \equiv 0$ .

We show the best possibility of the constant 4 in (2.1). We consider the case (1). Let us fix  $a > 0$ . Let  $\epsilon > 0$  be a small parameter and put

$$v^\epsilon(x) = \begin{cases} 0, & 0 \leq x < a, \\ (x - a)\psi(x)^{-1/2-\epsilon}, & a < x < a + 1, \\ \psi(x)^{-1/2-\epsilon}, & a + 1 < x. \end{cases} \quad (2.4)$$

Then we have after straightforward computations that

$$\begin{aligned} \frac{\int_0^\infty (v_x^\epsilon)^2 \psi^2 / \psi_x dx}{\int_0^\infty (v^\epsilon)^2 \psi_x dx} &= \frac{O(1) + (1/2 + \epsilon)^2 \frac{1}{2\epsilon} \psi(a + 1)^{-2\epsilon}}{O(1) + \frac{1}{2\epsilon} \psi(a + 1)^{-2\epsilon}} \\ &= \frac{O(\epsilon) + (1/2 + \epsilon)^2 \psi(a + 1)^{-2\epsilon}}{O(\epsilon) + \psi(a + 1)^{-2\epsilon}} \rightarrow \frac{1}{4} \end{aligned}$$

for  $\epsilon \rightarrow 0$ . This shows that 4 in (2.1) is the best possible constant. The case (2) can be treated similarly if we take a test function  $v^\epsilon(x)$  as

$$v^\epsilon(x) = \begin{cases} 0, & 0 \leq x < a, \\ (x - a)(-\psi(x))^{-1/2-\epsilon}, & a < x < a + 1, \\ (-\psi(x))^{-1/2-\epsilon}, & a + 1 < x, \end{cases}$$

but we omit the details. This completes the proof of Proposition 2.1.  $\square$

The  $L^p$  version of Proposition 2.1 is given as follows.

**Proposition 2.2.** *Let  $\psi$  be the same as in Proposition 2.1. Let  $1 < p < \infty$ . Then we have*

$$\int_0^\infty |v|^p \psi_x dx \leq p^p \int_0^\infty |v_x|^p |\psi|^p / \psi_x^{p-1} dx \quad (2.5)$$

for  $v \in C_0^\infty(\mathbb{R}_+)$  and hence for  $v \in W_0^{1,p}(w)$  with  $w = |\psi|^p / \psi_x^{p-1}$ . Here  $p^p$  is the best possible constant, and there is no function  $v \in W_0^{1,p}(w)$ ,  $v \neq 0$ , which attains the equality in (2.5).

*Proof.* We only prove the inequality (2.5) and omit the other discussions. Let  $1 < p < \infty$  and  $v \in C_0^\infty(\mathbb{R}_+)$ . A simple calculation gives

$$\begin{aligned} (|v|^p \psi)_x &= |v|^p \psi_x + p|v|^{p-2} v v_x \psi \\ &= \frac{1}{p} (|v|^p \psi_x - p^p |v_x|^p |\psi|^p / \psi_x^{p-1}) + R, \end{aligned} \quad (2.6)$$

where

$$R = \left(1 - \frac{1}{p}\right) |v|^p \psi_x + \frac{1}{p} p^p |v_x|^p |\psi|^p / \psi_x^{p-1} + p|v|^{p-2} v v_x \psi.$$

Integrating (2.6) in  $x$ , we obtain

$$\int_0^\infty |v|^p \psi_x dx + p \int_0^\infty R dx = p^p \int_0^\infty |v_x|^p |\psi|^p / \psi_x^{p-1} dx. \quad (2.7)$$

By applying the Young inequality  $AB \leq (1 - 1/p)A^{p/(p-1)} + (1/p)B^p$  for  $A = |v|^{p-1} \psi_x^{(p-1)/p}$  and  $B = p|v_x| |\psi| / \psi_x^{(p-1)/p}$ , we find that  $R \geq 0$ , which together with (2.7) gives the desired inequality (2.5).  $\square$

The following variant of Proposition 2.1 is useful in our application.

**Proposition 2.3.** *Let  $\phi \in C^1[0, \infty)$ ,  $\phi < 0$ ,  $\phi_x > 0$ , and  $\phi(x) \rightarrow 0$  for  $x \rightarrow \infty$ . Let  $\sigma \in \mathbb{R}$  with  $\sigma \neq 0$ , and define the weight functions  $w$  and  $w_1$  by*

$$w = (-\phi)^{-\sigma+1} / \phi_x, \quad w_1 = (-\phi)^{-\sigma-1} \phi_x. \quad (2.8)$$

Then we have

$$\int_0^\infty v^2 w_1 dx \leq \frac{4}{\sigma^2} \int_0^\infty v_x^2 w dx \quad (2.9)$$

for  $v \in H_0^1(w)$ . Here  $4/\sigma^2$  is the best possible constant, and there is no function  $v \in H_0^1(w)$ ,  $v \neq 0$ , which attains the equality in (2.9).

*Proof.* We put  $\psi = (-\phi)^{-\sigma}$  for  $\sigma > 0$  and  $\psi = -(-\phi)^{-\sigma}$  for  $\sigma < 0$ , and apply Proposition 2.1. This gives the desired conclusion.  $\square$

As a simple corollary of Proposition 2.3, we have:

**Corollary 2.4.** *Let  $\alpha \in \mathbb{R}$  with  $\alpha \neq 1$ . Then we have*

$$\|v\|_{L_{\alpha-2}^2} \leq \frac{2}{|\alpha-1|} \|v_x\|_{L_\alpha^2} \quad (2.10)$$

for  $v \in H_{\alpha,0}^1$ . Here the constant  $2/|\alpha-1|$  is the best possible, and there is no function  $v \in H_{\alpha,0}^1$ ,  $v \neq 0$ , which attains the equality in (2.10).

*Proof.* Let  $\phi = -(1+x)^{-1/q}$  with  $q > 0$ . We apply Proposition 2.3 for this  $\phi$  and  $\sigma = (\alpha-1)q$ . This gives the proof.  $\square$

### 3 Dissipativity of the linearized operator

Following [2], we discuss the dissipativity of the operator  $L$  defined by (1.5) in the weighted space  $L^2(w)$ , where  $w$  is given by (2.8) with  $\phi$  being the degenerate stationary wave. Note that our degenerate stationary wave  $\phi$  is a smooth solution of (1.3) and verifies

$$-1 \leq \phi(x) < 0, \quad \phi_x(x) > 0, \quad \phi(x) \rightarrow 0 \text{ for } x \rightarrow \infty, \quad (3.1)$$

$$c(1+x)^{-1/q} \leq -\phi(x) \leq C(1+x)^{-1/q}. \quad (3.2)$$

Now, letting  $w > 0$  be a smooth weight function depending only on  $x$ , we calculate the inner product  $\langle Lv, v \rangle_{L^2(w)}$  for  $v \in C_0^\infty(\mathbb{R}_+)$ , where

$$\langle u, v \rangle_{L^2(w)} := \int_0^\infty uvw \, dx. \quad (3.3)$$

We multiply (1.5) by  $v$ . Then a simple computation gives

$$(Lv)v = (vv_x - \frac{1}{2}f'(\phi)v^2)_x - v_x^2 - \frac{1}{2}f''(\phi)\phi_x v^2.$$

Multiplying this equality by  $w$ , we obtain

$$\begin{aligned} (Lv)vw &= \left\{ (vv_x - \frac{1}{2}f'(\phi)v^2)w - \frac{1}{2}v^2w_x \right\}_x \\ &\quad - v_x^2w + \frac{1}{2}v^2(w_{xx} + w_x f'(\phi) - w f''(\phi)\phi_x). \end{aligned} \quad (3.4)$$

Now we choose the weight function  $w$  and the corresponding  $w_1$  in terms of our degenerate stationary wave  $\phi$  by (2.8), where  $\sigma \in \mathbb{R}$ . Then we have  $w = (-\phi)^{-\sigma+1}/f(\phi)$  and  $w_1 = (-\phi)^{-\sigma-1}f(\phi)$  by  $\phi_x = f(\phi)$ . After straightforward computations, we find that

$$w_{xx} + w_x f'(\phi) - w f''(\phi)\phi_x = 2(c_1(\sigma) - r(\phi))w_1, \quad (3.5)$$

where

$$\begin{aligned} c_1(\sigma) &:= \sigma(\sigma-1)/2 - q(q+1), \\ r(u) &:= (-u)^2 f''(u)/f(u) - q(q+1). \end{aligned} \quad (3.6)$$

Substituting (3.5) into (3.4) and integrating with respect to  $x$ , we get the following conclusion.

**Claim 3.1.** *Let  $\phi$  be the degenerate stationary wave and define the weight functions  $w$  and  $w_1$  by (2.8) with  $\sigma \in \mathbb{R}$ . Then the operator  $L$  defined in (1.5) verifies*

$$\langle Lv, v \rangle_{L^2(w)} = -\|v_x\|_{L^2(w)}^2 + c_1(\sigma)\|v\|_{L^2(w_1)}^2 - \int_0^\infty v^2 r(\phi) w_1 dx \quad (3.7)$$

for  $v \in C_0^\infty(\mathbb{R}_+)$  and hence for  $v \in H_0^1(w)$ , where  $c_1(\sigma)$  and  $r(\phi)$  are given in (3.6).

The term  $r(\phi)$  in (3.7) can be regarded as a small perturbation. In fact, a straightforward computation gives

$$r(u) = (-u)\{(-u)g''(u) - 2(q+1)g'(u)\}/(1+g(u)), \quad (3.8)$$

which shows that  $r(u) = O(|u|)$  for  $u \rightarrow 0$ . In particular, we have  $r(u) \equiv 0$  if  $g(u) \equiv 0$ . With these preparations, we have the following result on the characterization of the dissipativity of  $L$ .

**Theorem 3.2.** *Assume (1.2). Let  $\phi$  be the degenerate stationary wave and  $L$  be the operator defined in (1.5). Let  $w$  and  $w_1$  be the weight functions in (2.8) with the parameter  $\sigma \in \mathbb{R}$ . Then we have:*

(1) *Let  $-2q < \sigma < 2(q+1)$ . Then, under the additional assumption that  $r(u) \geq 0$  for  $-1 \leq u \leq 0$ , the operator  $L$  is uniformly dissipative in  $L^2(w)$ . Namely, there is a positive constant  $\delta$  such that*

$$\langle Lv, v \rangle_{L^2(w)} \leq -\delta(\|v_x\|_{L^2(w)}^2 + \|v\|_{L^2(w_1)}^2) \quad \text{for } v \in H_0^1(w). \quad (3.9)$$

(2) *Let  $\sigma > 2(q+1)$  or  $\sigma < -2q$ . Then the operator  $L$  can not be dissipative in  $L^2(w)$ . Namely, we have  $\langle Lv, v \rangle_{L^2(w)} > 0$  for some  $v \in H_0^1(w)$  with  $v \neq 0$ .*

*Proof.* The proof is based on the equality (3.7) in Claim 3.1 and the Hardy type inequality (2.9) in Proposition 2.3.

Let  $-2q < \sigma < 2(q+1)$ . This is equivalent to  $c_1(\sigma) < \sigma^2/4$ . Therefore we can choose  $\delta > 0$  so small that  $\delta(1 + \sigma^2/4) \leq \sigma^2/4 - c_1(\sigma)$ . Since  $r(\phi) \geq 0$  by the additional assumption on  $r(u)$  and since  $(\sigma^2/4)\|v\|_{L^2(w_1)}^2 \leq \|v_x\|_{L^2(w)}^2$  by the Hardy type inequality (2.9), we have from (3.7) that

$$\begin{aligned} \langle Lv, v \rangle_{L^2(w)} &\leq -\|v_x\|_{L^2(w)}^2 + c_1(\sigma)\|v\|_{L^2(w_1)}^2 \\ &= -\delta\|v_x\|_{L^2(w)}^2 - (1-\delta)\|v_x\|_{L^2(w)}^2 + c_1(\sigma)\|v\|_{L^2(w_1)}^2 \\ &\leq -\delta\|v_x\|_{L^2(w)}^2 - \{(1-\delta)\sigma^2/4 - c_1(\sigma)\}\|v\|_{L^2(w_1)}^2 \\ &\leq -\delta(\|v_x\|_{L^2(w)}^2 + \|v\|_{L^2(w_1)}^2) \end{aligned} \quad (3.10)$$



for  $v \in C_0^\infty(\mathbb{R}_+)$  and hence for  $v \in H_0^1(w)$ , where we have used the fact that  $(1 - \delta)\sigma^2/4 - c_1(\sigma) \geq \delta$ . This completes the proof of the uniform dissipative case (1).

Next we consider the case where  $\sigma > 2(q + 1)$ ; the case  $\sigma < -2q$  can be treated similarly and we omit the argument in this latter case. When  $\sigma > 2(q + 1)$ , we have  $c_1(\sigma) > \sigma^2/4$ . Then we choose  $\delta > 0$  so small that  $c_1(\sigma) \geq \sigma^2/4 + 3\delta$ . Since  $r(u) = O(|u|)$  for  $u \rightarrow 0$  and  $\phi(x) \rightarrow 0$  for  $x \rightarrow \infty$ , we take  $a = a(\delta) > 0$  so large that  $|r(\phi)| \leq \delta$  for  $x \geq a$ . For this choice of  $a$  and for  $\epsilon > 0$ , we take a test function  $v^\epsilon$  as in (2.4):

$$v^\epsilon(x) = \begin{cases} 0, & 0 \leq x < a, \\ (x - a)(-\phi(x))^{\sigma(1/2+\epsilon)}, & a < x < a + 1, \\ (-\phi(x))^{\sigma(1/2+\epsilon)}, & a + 1 < x. \end{cases} \quad (3.11)$$

Then we have

$$\left| \int_0^\infty (v^\epsilon)^2 r(\phi) w_1 dx \right| \leq \delta \int_a^\infty (v^\epsilon)^2 w_1 dx = \delta \|v^\epsilon\|_{L^2(w_1)}^2,$$

so that we have from (3.7) that

$$\langle Lv^\epsilon, v^\epsilon \rangle_{L^2(w)} \geq -\|v_x^\epsilon\|_{L^2(w)}^2 + (c_1(\sigma) - \delta) \|v^\epsilon\|_{L^2(w_1)}^2. \quad (3.12)$$

Also, by straightforward computations, we find that

$$\begin{aligned} \frac{\|v_x^\epsilon\|_{L^2(w)}^2}{\|v^\epsilon\|_{L^2(w_1)}^2} &= \frac{O(1) + \sigma^2(1/2 + \epsilon)^2 \frac{1}{2\sigma\epsilon} (-\phi(a + 1))^{2\sigma\epsilon}}{O(1) + \frac{1}{2\sigma\epsilon} (-\phi(a + 1))^{2\sigma\epsilon}} \\ &= \frac{O(\epsilon) + \sigma^2(1/2 + \epsilon)^2 (-\phi(a + 1))^{2\sigma\epsilon}}{O(\epsilon) + (-\phi(a + 1))^{2\sigma\epsilon}} \rightarrow \frac{\sigma^2}{4} \end{aligned}$$

for  $\epsilon \rightarrow 0$ . Thus we have  $\|v_x^\epsilon\|_{L^2(w)}^2 / \|v^\epsilon\|_{L^2(w_1)}^2 \leq \sigma^2/4 + \delta$  for a suitably small  $\epsilon = \epsilon(\delta) > 0$ . Consequently, we have from (3.12) that

$$\begin{aligned} \frac{\langle Lv^\epsilon, v^\epsilon \rangle_{L^2(w)}}{\|v^\epsilon\|_{L^2(w_1)}^2} &\geq -\frac{\|v_x^\epsilon\|_{L^2(w)}^2}{\|v^\epsilon\|_{L^2(w_1)}^2} + c_1(\sigma) - \delta \\ &\geq -(\sigma^2/4 + \delta) + c_1(\sigma) - \delta \geq \delta. \end{aligned}$$

This completes the proof of the non-dissipative case (2). Thus the proof of Theorem 3.2 is complete.  $\square$

In the special case where  $g(u) \equiv 0$  so that  $f(u) = \frac{1}{q}(-u)^{q+1}$ , we have  $\phi = -(1+x)^{-1/q}$  and the operator  $L$  in (1.5) is reduced to

$$L_0 v = v_{xx} + \frac{q+1}{q} \left( \frac{v}{1+x} \right)_x. \quad (3.13)$$

In this simplest case, we have the complete characterization of the dissipativity of the operator  $L_0$ .

**Theorem 3.3.** *Let  $\alpha_c(q) := 3 + 2/q$ . Then we have the complete characterization of the dissipativity of the operator  $L_0$  given in (3.13):*

(1) *Let  $-1 < \alpha < \alpha_c(q)$ . Then  $L_0$  is uniformly dissipative in  $L_\alpha^2$ . Namely, there is a positive constant  $\delta$  such that*

$$\langle L_0 v, v \rangle_{L_\alpha^2} \leq -\delta (\|v_x\|_{L_\alpha^2}^2 + \|v\|_{L_{\alpha-2}^2}^2) \quad \text{for } v \in H_{\alpha,0}^1. \quad (3.14)$$

(2) *Let  $\alpha = \alpha_c(q)$  or  $\alpha = -1$ . Then  $L_0$  is strictly dissipative in  $L_\alpha^2$ . Namely, we have  $\langle L_0 v, v \rangle_{L_\alpha^2} < 0$  for  $v \in H_{\alpha,0}^1$  with  $v \neq 0$ .*

(3) *Let  $\alpha > \alpha_c(q)$  or  $\alpha < -1$ . Then  $L_0$  can not be dissipative in  $L_\alpha^2$ . Namely, we have  $\langle L_0 v, v \rangle_{L_\alpha^2} > 0$  for some  $v \in H_{\alpha,0}^1$  with  $v \neq 0$ .*

*Proof.* In this case, we have  $\phi = -(1+x)^{-1/q}$ ,  $L = L_0$  and  $r(u) \equiv 0$ . Therefore, (3.7) is reduced to

$$\langle L_0 v, v \rangle_{L^2(w)} = -\|v_x\|_{L^2(w)}^2 + c_1(\sigma) \|v\|_{L^2(w_1)}^2, \quad (3.15)$$

where  $w$  and  $w_1$  are the weight functions defined in (2.8) with  $\phi = -(1+x)^{-1/q}$  and  $\sigma = (\alpha - 1)q$ . The desired conclusions easily follow from (3.15) by applying the same argument as in Theorem 3.2. We omit the details.  $\square$

## 4 Nonlinear stability

The following stability result for the nonlinear problem (1.4) was obtained in [2] as a refinement of the result in [14].

**Theorem 4.1.** *Assume (1.2). Suppose that  $v_0 \in L_\alpha^2 \cap L^\infty$  for some  $\alpha$  with  $1 \leq \alpha < \alpha_c(q) := 3 + q/2$ . Then there is a positive constant  $\delta_1$  such that if  $\|v_0\|_{L_1^2} \leq \delta_1$ , then the problem (1.4) has a unique global solution  $v \in C^0([0, \infty); L_\alpha^2 \cap L^p)$  for each  $p$  with  $2 \leq p < \infty$ . Moreover, the solution verifies the decay estimate*

$$\|v(t)\|_{L^p} \leq C(\|v_0\|_{L_\alpha^2} + \|v_0\|_{L^\infty})(1+t)^{-\alpha/4-\nu} \quad (4.1)$$

for  $t \geq 0$ , where  $2 \leq p < \infty$ ,  $\nu = (1/2)(1/2 - 1/p)$ , and  $C$  is a positive constant.

*Proof.* A key to the proof of this theorem is to show the following space-time weighted energy inequality:

$$\begin{aligned} (1+t)^\gamma \|v(t)\|_{L_\beta^2}^2 + \int_0^t (1+\tau)^\gamma (\|v_x(\tau)\|_{L_\beta^2}^2 + \|v(\tau)\|_{L_{\beta-2}^2}^2) d\tau \\ \leq C \|v_0\|_{L_\beta^2}^2 + \gamma C \int_0^t (1+\tau)^{\gamma-1} \|v(\tau)\|_{L_\beta^2}^2 d\tau + CS_\beta^\gamma(t) \end{aligned} \quad (4.2)$$

for any  $\gamma \geq 0$  and  $\beta$  with  $0 \leq \beta \leq \alpha$ , where  $1 \leq \alpha < \alpha_c(q) := 3 + 2/q$ ,  $C$  is a constant independent of  $\gamma$  and  $\beta$ , and

$$S_\beta^\gamma(t) = \int_0^t (1+\tau)^\gamma \|v(\tau)\|_{L_{\beta-1}^3}^3 d\tau. \quad (4.3)$$

Here we give an outline of the proof of (4.2) and omit the other discussions. We refer to [2, 14] for the complete proof of Theorem 4.1.

*Proof of (4.2) for  $\beta = 0$ .* The proof is based on the time weighted  $L^2$  energy method. First we note that

$$\|v(t)\|_{L^\infty} \leq M_\infty, \quad (4.4)$$

where  $M_\infty = \|v_0\|_{L^\infty} + 2$ . This is an easy consequence of the maximum principle (see [5] for the details). Now we multiply the equation (1.4) by  $v$ . This yields

$$\left(\frac{1}{2}v^2\right)_t + (F - vv_x)_x + v_x^2 + G = 0, \quad (4.5)$$

where

$$\begin{aligned} F &= (f(\phi + v) - f(\phi))v - \int_0^v (f(\phi + \eta) - f(\phi))d\eta, \\ G &= \int_0^v (f'(\phi + \eta) - f'(\phi))d\eta \cdot \phi_x. \end{aligned} \quad (4.6)$$

We note that

$$F = \frac{1}{2}f'(\phi)v^2 + O(|v|^3), \quad G = \frac{1}{2}f''(\phi)\phi_x v^2 + \phi_x O(|v|^3) \quad (4.7)$$

for  $v \rightarrow 0$ . Here, a careful computation, using (3.2) and (4.4), shows that

$$G \geq c(1+x)^{-2}v^2 - C(1+x)^{-1-1/q}|v|^3 \quad (4.8)$$

for any  $x \in \mathbb{R}_+$ . We integrate (4.5) over  $\mathbb{R}_+$  and substitute (4.8) into the resulting equality, obtaining

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 + c \|v\|_{L_{-2}^2}^2 \leq C \|v\|_{L_{-1}^3}^3.$$

We multiply this inequality by  $(1+t)^\gamma$  and integrate with respect  $t$ . This yields the desired inequality (4.2) for  $\beta = 0$ .

*Proof of (4.2) for  $\beta > 0$ .* We apply the space-time weighted energy method employed in [14, 2] (see also [3]). Let  $w > 0$  be a smooth weight function depending only on  $x$ , which will be specified later. We multiply (4.5) by  $w$ , obtaining

$$\begin{aligned} \left(\frac{1}{2}v^2w\right)_t + \left\{(F - \mu vv_x)w + \frac{1}{2}v^2w_x\right\}_x \\ + v_x^2w - \left(\frac{1}{2}v^2w_{xx} + Fw_x - Gw\right) = 0. \end{aligned} \quad (4.9)$$

Here, using (4.7), we have

$$\frac{1}{2}v^2w_{xx} + Fw_x - Gw = \frac{1}{2}v^2(w_{xx} + w_x f'(\phi) - w f''(\phi)\phi_x) + R, \quad (4.10)$$

where  $R = w_x O(|v|^3) - w \phi_x O(|v|^3)$  for  $v \rightarrow 0$ . Notice that the coefficient  $w_{xx} + w_x f'(\phi) - w f''(\phi)\phi_x$  in (4.10) is just the same as that appeared in (3.4). Now we choose the weight function  $w$  and the corresponding  $w_1$  by (2.8) with  $\sigma = (\beta - 1)q$ , where  $0 \leq \beta \leq \alpha$  and  $1 \leq \alpha < \alpha_c(q) := 3 + 2/q$ . Then we have (3.5) with  $\sigma = (\beta - 1)q$ . Substituting these expressions into (4.9) and integrating over  $\mathbb{R}_+$ , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{L^2(w)}^2 + \|v_x\|_{L^2(w)}^2 - c_1(\sigma) \|v\|_{L^2(w_1)}^2 \\ + \int_0^\infty v^2 r(\phi) w_1 dx = \int_0^\infty R dx, \end{aligned} \quad (4.11)$$

where  $c_1(\sigma)$  and  $r(\phi)$  are given in (3.6) with  $\sigma = (\beta - 1)q$ . Here our weight functions verify

$$w \sim (1+x)^\beta, \quad w_1 \sim (1+x)^{\beta-2}, \quad (4.12)$$

where the symbol  $\sim$  means the equivalence. This implies that the norms  $\|\cdot\|_{L^2(w)}$  and  $\|\cdot\|_{L^2(w_1)}$  are equivalent to  $\|\cdot\|_{L^\beta_\beta}$  and  $\|\cdot\|_{L^{\beta-2}_\beta}$ , respectively.

We estimate (4.11) similarly as in (1) of Theorem 3.2. To this end, we note that  $\sigma_1 \leq \sigma \leq \sigma_2$ , where  $\sigma_1 = -q$  and  $\sigma_2 = (\alpha - 1)q$ . Since  $c_1(\sigma) < \sigma^2/4$  for  $-2q < \sigma < 2(q+1)$  and since  $-2q < \sigma_1 < \sigma_2 < 2(q+1)$ , we can choose  $\delta > 0$  so small that

$$\delta \leq \min_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{\sigma^2/4 - c_1(\sigma)}{2 + \sigma^2/4}.$$

Notice that this  $\delta$  is independent of  $\beta$ . For this choice of  $\delta$ , we take  $a = a(\delta) > 0$  so large that  $|r(\phi)| \leq \delta$  for  $x \geq a$ . Then we have

$$\left| \int_0^\infty v^2 r(\phi) w_1 dx \right| \leq \delta \|v\|_{L^2(w_1)}^2 + C \|v\|_{L^2_{-2}}^2,$$

where  $C$  is a constant satisfying  $C \geq (1+x)^2 |r(\phi)| w_1$  for  $0 \leq x \leq a$ . Also, using the Hardy type inequality  $(\sigma^2/4) \|v\|_{L^2(w_1)}^2 \leq \|v_x\|_{L^2(w)}^2$  in (2.9) and estimating similarly as in (3.10), we have

$$\|v_x\|_{L^2(w)}^2 - c_1(\sigma) \|v\|_{L^2(w_1)}^2 \geq \delta \|v_x\|_{L^2(w)}^2 + 2\delta \|v\|_{L^2(w_1)}^2,$$

where we have used the fact that  $(1-\delta)\sigma^2/4 - c_1(\sigma) \geq 2\delta$ . On the other hand, using (4.4), we see that  $|R| \leq C(|w_x| + w\phi_x)|v|^3$ . Moreover, a straightforward computation shows that  $|w_x| + w\phi_x \leq C(1+x)^{\beta-1}$ . Substituting all these estimates into (4.11), we obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2(w)}^2 + \delta (\|v_x\|_{L^2(w)}^2 + \|v\|_{L^2(w_1)}^2) \leq C \|v\|_{L^2_-}^2 + C \|v\|_{L^3_{\beta-1}}^3, \quad (4.13)$$

where  $\delta$  and  $C$  are independent of  $\beta$ . We multiply this inequality by  $(1+t)^\gamma$  and integrate with respect to  $t$ . By virtue of (4.12), we have

$$\begin{aligned} & (1+t)^\gamma \|v(t)\|_{L^2_\beta}^2 + \int_0^t (1+\tau)^\gamma (\|v_x(\tau)\|_{L^2_\beta}^2 + \|v(\tau)\|_{L^2_{\beta-2}}^2) d\tau \\ & \leq C \|v_0\|_{L^2_\beta}^2 + \gamma C \int_0^t (1+\tau)^{\gamma-1} \|v(\tau)\|_{L^2_\beta}^2 d\tau \\ & \quad + C \int_0^t (1+\tau)^\gamma \|v(\tau)\|_{L^2_-}^2 d\tau + C S_\beta^\gamma(t), \end{aligned} \quad (4.14)$$

where the constant  $C$  is independent of  $\gamma$  and  $\beta$ . Here the third term on the right hand side of (4.14) was already estimated by (4.2) with  $\beta = 0$ . Hence we have proved (4.2) also for  $0 < \beta \leq \alpha$ . This completes the proof.  $\square$

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